On the global Gaussian Lipschitz space

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Abstract. A Lipschitz space is defined in the Ornstein-Uhlenbeck setting, by means of a bound for the gradient of the Ornstein-Uhlenbeck Poisson integral. This space is then characterized with a Lipschitz-type continuity condition. These functions turn out to have at most logarithmic growth at infinity. The analogous Lipschitz space containing only bounded functions was introduced by Gatto and Urbina and has been characterized by the authors in [4].

1 Introduction and main result

Consider the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian measure γ, given by

$$d\gamma(x) = \pi^{-n/2} e^{-|x|^2}.$$

The Gaussian analogue of the Euclidean Laplacian is the *Ornstein-Uhlenbeck operator*

$$\mathcal{L} = -\frac{1}{2}\Delta + x \cdot \nabla,$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$. The heat semigroup generated by $\mathcal{L}$ and defined in $L^2(\gamma)$ is the so-called *Ornstein-Uhlenbeck semigroup*

$$T_t = e^{-t\mathcal{L}}, \quad t \ge 0.$$

The *Ornstein-Uhlenbeck Poisson semigroup* $P_t = e^{-t\sqrt{-\mathcal{L}}}$, $t \ge 0$, can be defined from $\{T_t\}_{t\ge 0}$ by subordination as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/(4u)} f(x)\, du, \qquad x \in \mathbb{R}^n,$$

for $f \in L^2(\gamma)$. As explained in Section 2, $P_t f$ is given by integration against a kernel $P_t(x,y)$.

Via $\{P_t\}_{t\ge 0}$, Gatto and Urbina [3] introduced the *Gaussian Lipschitz space* GLip_α for all $\alpha > 0$. We shall always have $\alpha \in (0,1)$. Then the definition says that a function f in $\mathbb{R}^n$ is in GLip_α if it is bounded and satisfies

$$\|t\partial_t P_t f\|_{L^\infty} \le A t^\alpha, \qquad t > 0, \tag{1.1}$$

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for some $A > 0$. These spaces and also Gaussian Besov spaces were studied in a series of works; see [2, 3, 5] and also the authors' paper [4].

In [4], the authors characterized GLip_α, $0 < \alpha < 1$, in terms of a Lipschitz-type continuity condition. Indeed, Theorem 1.1 of [4] says that $f \in \mathrm{GLip}_\alpha$ if and only if there exists a positive constant K such that

$$|f(x) - f(y)| \le K \min\left\{|x-y|^\alpha,\ \left(\frac{|x-y_x|}{1+|x|}\right)^{\frac{\alpha}{2}} + |y_x'|^\alpha\right\}, \qquad x,\, y \in \mathbb{R}^n. \tag{1.2}$$

Here and in what follows, we use a decomposition of y as $y = y_x + y_x'$, where y_x is parallel to x and y_x' orthogonal to x; however, if $x = 0$ or $n = 1$, we let $y_x = y$ and $y_x' = 0$.

As is well known, a condition analogous to (1.1) for the standard Poisson integral characterizes the ordinary Lipschitz space; see [6, Sect. V.4]. If only bounded functions are considered, one obtains the inhomogeneous Lipschitz space, and without the boundedness assumption the larger, homogeneous Lipschitz space.

In our setting, we shall see that the condition (1.1) without the boundedness condition defines a Gaussian analogue of the homogeneous Lipschitz space. Since here no homogeneity is involved, we shall call it the *global Gaussian Lipschitz space.*

In (1.1), an a priori assumption is needed to assure that $P_t f$ exists. Here we apply a recent result by Garrigós, Harzstein, Signes, Torrea and Viviani [1]. Clearly, a measurable function f in $\mathbb{R}^n$ has a well-defined Gaussian Poisson integral if

$$\int P_t(x,y)|f(y)|\,dy < \infty,$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Theorem 1.1 of [1] says that this is equivalent to the growth condition

$$\int_{\mathbb{R}^n} \frac{e^{-|y|^2}}{\sqrt{\ln(e+|y|)}}\,|f(y)|\,dy < \infty. \tag{1.3}$$

Moreover, (1.3) ensures that $P_t f(x) \to f(x)$ as $t \to 0$ for a.a. $x \in \mathbb{R}^n$.

We can now define the global Gaussian Lipschitz space.

Definition 1.1. *Let $\alpha \in (0,1)$. A measurable function f defined in $\mathbb{R}^n$ and satisfying* (1.3) *belongs to the global Gaussian Lipschitz space* GGLip_α *if* (1.1) *holds. The corresponding norm is*

$$\|f\|_{\mathrm{GGLip}_\alpha} = \inf\{A > 0 :\ A \text{ satisfies } (1.1)\}.$$

Strictly speaking, this space consists of functions modulo constants. A natural question is now what continuity condition characterizes this space. To state the answer, we start in one dimension and introduce a distance by

$$d(x,y) = \left|\int_x^y \frac{d\xi}{1+|\xi|}\right|, \qquad x, y \in \mathbb{R}. \tag{1.4}$$

Then

$$d(x,y) = |\ln(1+|x|) - \operatorname{sgn} xy\ \ln(1+|y|)|$$

for all $x, y \in \mathbb{R}$, provided we define $\operatorname{sgn} 0 = 1$. In several dimensions, we use this distance on the line spanned by x, defining

$$d(x, y_x) = \left| \ln(1+|x|) - \operatorname{sgn}\langle x, y\rangle \ln(1+|y_x|) \right|, \qquad x, y \in \mathbb{R}^n,$$

with y_x as before.

Our result reads as follows.

Theorem 1.2. *Let $\alpha \in (0,1)$ and let f be a measurable function in $\mathbb{R}^n$. The following are equivalent:*

(i) *f satisfies* (1.3) *and $f \in \mathrm{GGLip}_\alpha$;*

(ii) *There exists a positive constant K such that*

$$|f(x) - f(y)| \le K \min\left\{ |x-y|^\alpha,\ d(x, y_x)^{\frac{\alpha}{2}} + |y'_x|^\alpha \right\}, \qquad x,\ y \in \mathbb{R}^n, \tag{1.5}$$

after correction of f on a null set.

Moreover,

$$\|f\|_{\mathrm{GGLip}_\alpha} \simeq \inf\{K :\ K \text{ satisfies } (1.5)\}. \tag{1.6}$$

The meaning of the symbol $\simeq$ is explained below.

Remark 1.3. To compare (1.5) and (1.2), one easily verifies that if $\langle x, y\rangle > 0$ and $1/2 < |x|/|y_x| < 2$, then

$$d(x, y_x) \simeq \frac{|x - y_x|}{1+|x|}. \tag{1.7}$$

Moreover, the space GLip_α can be described in terms of the distance function d. Indeed, as (1.2) implies boundedness (see [4, Lemma 2.1]), it is easy to check that (1.2) holds if and only if there exists a constant $K' > 0$ such that

$$|f(x) - f(y)| \le K' \min\left\{ 1,\ |x-y|^\alpha,\ d(x, y_x)^{\frac{\alpha}{2}} + |y'_x|^\alpha \right\}$$

for all $x,\ y \in \mathbb{R}^n$. This also tells us that for bounded functions, (1.2) is equivalent to (1.5). But (1.5) implies only that

$$f(x) = O((\ln|x|)^{\alpha/2}) \qquad \text{as} \qquad |x| \to \infty.$$

This condition is sharp, as shown by an example in Section 5; observe that it is much stronger than (1.3).

The paper is organized as follows. Section 2 contains a needed improvement of the estimate for $P_t(x, y)$ and its derivatives in [4]. Some properties of the Gaussian Poisson integral are obtained in Section 3. Then Theorem 1.2 is proved in Section 4. Finally, we give in Section 5 an example of a function in GGLip_α with logarithmic growth.

Notation. Throughout the paper, we shall write C for various positive constants which depend only on n and α, unless otherwise explicitly stated. Given any two nonnegative quantities A and B, the notation $A \lesssim B$ stands for $A \le CB$ (we say that A is controlled by B), and $A \gtrsim B$ means $B \lesssim A$. If $B \lesssim A \lesssim B$, we write $A \simeq B$.

For positive quantities X, we shall write $\exp^*(-X)$, meaning $\exp(-cX)$ for some constant $c = c(n, \alpha) > 0$.

2 The Ornstein-Uhlenbeck Poisson kernel

It is known that for $f \in L^2(\gamma)$,

$$T_t f(x) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} M_{e^{-t}}(x,y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \;\; t > 0,$$

where $M_{e^{-t}}$ is the *Mehler kernel* defined by

$$M_r(x,y) = \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{n/2}}, \qquad x,\, y \in \mathbb{R}^n, \quad 0 < r < 1.$$

The Gaussian Poisson integral $P_t f$ is given by an integral kernel called the *Ornstein-Uhlenbeck Poisson kernel* and denoted by $P_t(x,y)$; thus

$$P_t f(x) = \int_{\mathbb{R}^n} P_t(x,y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \;\; t > 0.$$

Because of the subordination formula, $P_t(x,y)$ is given by

$$\begin{aligned} P_t(x,y) &= \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, M_{e^{-t^2/(4u)}}(x,y)\, du \\ &= \frac{1}{2\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{\exp(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}})}{(1-e^{-2s})^{n/2}}\, ds. \end{aligned} \tag{2.1}$$

Here we inserted the expression for the Mehler kernel and transformed the variable.

The following estimate for P_t and its first derivatives is established in [4, Theorems 1.2 and 1.3].

Proposition 2.1. *For all $t > 0$, $x,\, y \in \mathbb{R}^n$ and $i \in \{1, 2, \dots, n\}$, the kernel P_t satisfies*

$$P_t(x,y) + |t\partial_t P_t(x,y)| + |t\partial_{x_i} P_t(x,y)| \le C\left[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)\right],$$

where

$$K_1(t,x,y) = \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}}\, \exp^*\left(-t(1+|x|)\right);$$

$$K_2(t,x,y) = \frac{t}{|x|}\left(t^2 + \frac{|x-y_x|}{|x|} + |y_x'|^2\right)^{-\frac{n+2}{2}} \exp^*\left(-\frac{(t^2+|y_x'|^2)|x|}{|x-y_x|}\right) \chi_{\{|x|>1,\; x\cdot y>0,\; |x|/2 \le |y_x| < |x|\}};$$

$$K_3(t,x,y) = \min(1,t)\, \exp^*(-|y|^2);$$

$$K_4(t,x,y) = \frac{t}{|y_x|}\left(\ln\frac{|x|}{|y_x|}\right)^{-\frac{3}{2}} \exp^*\left(-\frac{t^2}{\ln\frac{|x|}{|y_x|}}\right) \exp^*(-|y_x'|^2)\, \chi_{\{x\cdot y>0,\; 1<|y_x|<|x|/2\}}.$$

We need a slight sharpening of this lemma. The term K_3 will be modified to decay for large x.

Lemma 2.2. *The estimate of Proposition 2.1 remains valid if the kernel $K_3(t,x,y)$ is replaced by*

$$\widetilde{K_3}(t,x,y) = \min\left\{1,\ \frac{t}{[\ln(e+|x|)]^{1/2}}\right\} \exp^*(-|y|^2).$$

Proof. From the proof of [4, Theorem 1.3], we see that $|t\partial_t P_t(x,y)|$ and $|t\partial_{x_i} P_t(x,y)|$ can be controlled by an integral similar to the right-hand side of (2.1) (only with exp in (2.1) replaced by $\exp^*$). Thus, we only need to consider $P_t(x,y)$.

When $|x| \le 4+2|y_1|$, we have $\exp^*(-|y|^2) \lesssim \exp^*(-|y|^2)\exp^*(-|x|^2)$ and hence $K_3(t,x,y) \lesssim \widetilde{K_3}(t,x,y)$.

Thus we assume from now on that $|x| > 4+2|y_1|$. We shall sharpen a few arguments in the proof of [4, Proposition 4.1]. By the rotation invariance of $P_t(x,y)$ and $\widetilde{K_3}(t,x,y)$, we may assume that $x = (x_1, 0, \ldots, 0)$ with $x_1 > 0$. The decomposition of y will then be written $y = (y_1, 0, \ldots, 0) + (0, y')$, and $|y_1| < x_1/2$.

Case 1. $-x_1/2 < y_1 \le 0$. Using the notation from the proof of [4, Proposition 4.1(i)], we see that we only need to verify that $J_2 \lesssim \widetilde{K_3}$. By [4, formula (4.9)] and the fact that $y_1 \le 0 < x_1$, we have

$$\begin{aligned} J_2 &\simeq \exp^*(-|y'|^2) \int_{\ln 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-|y_1 - e^{-s}x_1|^2)\, ds \\ &\lesssim \exp^*(-|y|^2) \int_{\ln 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-e^{-2s}x_1^2)\, ds. \end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned} \int_{\frac12 \ln x_1}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-e^{-2s}x_1^2)\, ds &\simeq \int_{\frac12 \ln x_1}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right)\, ds \\ &\lesssim \min\left\{1,\ t(\ln x_1)^{-1/2}\right\} \end{aligned}$$

and

$$\begin{aligned} \int_{\ln 2}^{\frac12 \ln x_1} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-e^{-2s}x_1^2)\, ds &\le \exp^*(-x_1) \int_{\ln 2}^{\frac12 \ln x_1} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right)\, ds \\ &\lesssim \exp^*(-x_1)\min\{1,\ t\}, \end{aligned}$$

from which the required estimate follows.

Case 2: $0 < y_1 < x_1/2$. Considering now the proof of [4, Proposition 4.1(iii)], we only need to estimate the terms $J_{2,1}^{(2)}$ and $J_{2,3}$, and also $J_{2,2}$ when $y_1 \in (0,1]$.

From [4, formula (4.16)], we get for $y_1 \in (0,1]$,

$$J_{2,2} \simeq \frac{t}{(\ln \frac{x_1}{y_1})^{3/2}} \exp^*\left(-\frac{t^2}{\ln \frac{x_1}{y_1}}\right) \exp^*\left(-|y'|^2\right)$$

$$\lesssim \min\left\{\frac{t}{(\ln\frac{x_1}{y_1})^{3/2}},\, \frac{1}{\ln\frac{x_1}{y_1}}\right\}\exp^*\left(-|y|^2\right)$$
$$\lesssim \widetilde{K_3}(t,x,y),$$

since here $\ln(x_1/y_1) \gtrsim \ln(e+|x|)$. Further,

$$J_{2,1}^{(2)} + J_{2,3} \le \exp^*(-|y'|^2)\int \frac{t}{s^{3/2}}\exp^*\left(-\frac{t^2}{s}\right)\exp^*(-|y_1 - e^{-s}x_1|^2)\,ds, \tag{2.3}$$

where the integral is taken over the set $\{s > \ln 2 : |s - \ln(x_1/y_1)| > c_0\}$, for some $c_0 > 0$. Thus the quotient $e^{-s}x_1/y_1$ stays away from 1 in this integral, so that $|y_1 - e^{-s}x_1| \simeq \max\{e^{-s}x_1,\, y_1\} \simeq e^{-s}x_1 + y_1$. This implies that the right-hand side of (2.3) is controlled by the expression in (2.2) and thus by $\widetilde{K_3}$.

Lemma 2.2 is proved. □

3 Auxiliary lemmas

Lemma 3.1. *There exists a constant $C > 0$ such that for all $x,\, y \in \mathbb{R}^n$ and $t > 0$,*

$$|\partial_t P_t(x,y)| \le C\,\frac{1}{t}\,P_{t/2}(x,y).$$

Proof. Differentiating (2.1), we get

$$\partial_t P_t(x,y) = \frac{1}{2\pi^{(n+1)/2}}\,\frac{1}{t}\int_0^\infty \frac{t}{s^{3/2}}\,e^{-\frac{t^2}{4s}}\left(1 - \frac{t^2}{2s}\right)\frac{e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{n/2}}\,ds.$$

It is now enough to observe that

$$e^{-\frac{t^2}{4s}}\left|1 - \frac{t^2}{2s}\right| \lesssim e^{-\frac{(t/2)^2}{4s}}$$

and compare with (2.1). □

Lemma 3.2. *Fix $i \in \{1,2,\dots,n\}$ and let $R > 0$. Then there exists a constant $C > 0$, depending only on n and R, such that for all $x,\, y \in \mathbb{R}^n$ with $|x| < R$,*

$$|\partial_{x_i} P_t(x,y)| \le C\,(1 + t^{-4-n})P_{t/2}(x,y), \qquad t > 0, \tag{3.1}$$

and

$$|\partial_{x_i} P_t(x,y)| \le C t^{-1/2} e^{-|y|^2}[\ln(e+|y|)]^{-3/4}, \qquad t > 1. \tag{3.2}$$

Proof. In this proof, all constants denoted C will depend only on n and R, and the same applies to the implicit constants in the $\lesssim$ and $\simeq$ symbols. We let $|x| < R$, and we can clearly assume that $R > 1$.

Differentiating (2.1), we get

$$\partial_{x_i} P_t(x,y) = \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{e^{-s}(y_i - e^{-s}x_i)}{1-e^{-2s}}\, \frac{\exp(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}})}{(1-e^{-2s})^{n/2}}\, ds. \tag{3.3}$$

Compared with (2.1), the integral has now an extra factor $e^{-s}(y_i - e^{-s}x_i)/(1-e^{-2s})$.

With $\gamma > 0$, we shall use repeatedly the simple inequality

$$e^{-\frac{t^2}{4s}} \le C_\gamma \left(\frac{s}{t^2}\right)^\gamma e^{-\frac{(t/2)^2}{4s}} \tag{3.4}$$

for some $C_\gamma > 0$, and here we sometimes drop the last factor.

We start with the simple case of bounded y; more precisely we assume $|y| \le e^{12} R$. Then the extra factor is no larger than $Ce^{-s}/(1-e^{-2s})$. An application of (3.4) with $\gamma = 1 + n/2$ yields

$$|\partial_{x_i} P_t(x,y)| \lesssim t^{-2-n} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{(t/2)^2}{4s}}\, \frac{e^{-s} s^{1+n/2}}{(1-e^{-2s})^{1+n/2}} \exp\left(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}\right) ds.$$

Comparing with (2.1), one sees that this estimate implies (3.1). If we choose instead $\gamma = 2+n/2$, (3.2) will also follow, since y stays bounded.

From now on, we assume that $|y| > e^{12} R$. Then (3.3) implies

$$|\partial_{x_i} P_t(x,y)| \lesssim \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{e^{-s}|y|}{1-e^{-2s}}\, \frac{\exp(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}})}{(1-e^{-2s})^{n/2}}\, ds. \tag{3.5}$$

We first estimate the exponent

$$E(s,x,y) = -\frac{|y-e^{-s}x|^2}{1-e^{-2s}}$$

from (3.5). It satisfies

$$E(s,x,y) \le \frac{-|y|^2 + 2e^{-s} y\cdot x}{1-e^{-2s}} \le \frac{-|y|^2 + \frac{1}{2}e^{-2s}|y|^2 + 2|x|^2}{1-e^{-2s}},$$

where we applied the inequality between the geometric and arithmetic means. If $e^{-s} < 1/2$, then

$$E(s,x,y) \le \frac{-|y|^2 + \frac{1}{2}e^{-2s}|y|^2}{1-e^{-2s}} + C.$$

If instead $e^{-s} \ge 1/2$, we have $2|x|^2 < e^{-2s}|y|^2/4$ since $|y| > e^{12}|x|$, and thus

$$E(s,x,y) \le \frac{-|y|^2 + \frac{3}{4}e^{-2s}|y|^2}{1-e^{-2s}}.$$

In both cases,

$$E(s,x,y) \le -|y|^2\, \frac{1-\frac{3}{4}e^{-2s}}{1-e^{-2s}} + C \le -|y|^2\left(1 + \frac{1}{4}e^{-2s}\right) + C,$$

and this implies

$$e^{E(s,x,y)} \lesssim e^{-|y|^2} \min\left(1, \frac{e^{2s}}{|y|^2}\right). \tag{3.6}$$

We also need a converse inequality, under the assumption that $s > \ln|y|$. Then

$$E(s,x,y) \geq \frac{-|y|^2 - 2e^{-s}|y||x| - e^{-2s}|x|^2}{1-e^{-2s}} \geq \frac{-|y|^2}{1-|y|^{-2}} - C \geq -|y|^2 - C. \tag{3.7}$$

Now split the integral in (3.5) as

$$\left(\int_0^3 + \int_3^{\ln|y|} + \int_{\ln|y|}^\infty\right) \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{e^{-s}|y|}{1-e^{-2s}}\, \frac{\exp(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}})}{(1-e^{-2s})^{n/2}}\, ds = I_1 + I_2 + I_3,$$

say; observe that $\ln|y| > 12$. We shall prove that these three integrals satisfy the bounds in (3.1) and (3.2).

In I_3, we have $e^{-s}|y|/(1-e^{-2s}) \lesssim 1$. Comparing with (2.1), we conclude that

$$I_3 \lesssim P_t(x,y) \lesssim P_{t/2}(x,y),$$

which is part of (3.1). Aiming at (3.2), we apply (3.4) with $\gamma = 3/4$ and (3.6), where the minimum is 1, to conclude that

$$I_3 \lesssim \int_{\ln|y|}^\infty t^{-1/2}\, s^{-3/4}\, e^{-s}\, |y|\, e^{-|y|^2}\, ds \lesssim t^{-1/2}\, (\ln|y|)^{-3/4}\, e^{-|y|^2},$$

as desired.

To deal with I_2, we apply (3.6), now with the second quantity in the minimum, and obtain

$$I_2 \lesssim \int_3^{\ln|y|} \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{e^s}{|y|}\, e^{-|y|^2}\, ds. \tag{3.8}$$

Using (3.4), again with $\gamma = 3/4$, we can estimate this integral by

$$t^{-1/2} e^{-|y|^2} \int_3^{\ln|y|} s^{-3/4}\, \frac{e^s}{|y|}\, ds,$$

which gives the bound in (3.2) for I_2. Thinking of (3.1), we write the integral in (3.8) as

$$te^{-|y|^2} |y|^{-1} \int_3^{\ln|y|} \phi(s) e^{s/2}\, ds,$$

where

$$\phi(s) = \frac{e^{s/2}}{s^{3/2}}\, e^{-\frac{t^2}{4s}}.$$

Here both the factors are increasing functions of s in $(3,\infty)$, and so is ϕ. Thus for any $\eta \in (0,1)$,

$$\sup_{(3,\ln|y|)} \phi(s) \leq \phi(\eta + \ln|y|),$$

and so

$$I_2 \lesssim te^{-|y|^2}|y|^{-1}\phi(\eta+\ln|y|)\int_3^{\ln|y|} e^{s/2}\,ds \simeq te^{-|y|^2}\frac{1}{(\eta+\ln|y|)^{3/2}}\,e^{-\frac{t^2}{4(\eta+\ln|y|)}}.$$

Integrating in η, we see that

$$I_2 \lesssim \int_{\ln|y|}^{1+\ln|y|} \frac{t}{s^{3/2}}\,e^{-\frac{t^2}{4s}}\,e^{-|y|^2}\,ds. \tag{3.9}$$

Because of (3.7), this integral is dominated by the one defining $P_t(x,y)$ in (2.1). Since $P_t(x,y) \lesssim P_{t/2}(x,y)$, it follows that $I_2 \lesssim P_{t/2}(x,y)$.

Finally, we estimate I_1 by means of (3.6). Since here $1-e^{-2s} \simeq s$, we get

$$I_1 \lesssim \int_0^3 \frac{t}{s^{3/2}}\,e^{-\frac{t^2}{4s}}\,\frac{1}{|y|\,s^{1+n/2}}\,e^{-|y|^2}\,ds.$$

Using (3.4) with $\gamma = 2+n/2$, we conclude that

$$I_1 \lesssim t^{-3-n}\int_0^3 s^{-1/2}e^{-\frac{(t/2)^2}{4s}}\,\frac{1}{|y|}\,e^{-|y|^2}\,ds. \tag{3.10}$$

This leads immediately to the bound in (3.2). For (3.1), we can estimate the right-hand side in (3.10) by

$$t^{-3-n}\frac{1}{|y|}\,e^{-\frac{t^2}{48}}\,e^{-|y|^2} \lesssim t^{-4-n}\,\frac{t}{(\eta+\ln|y|)^{3/2}}\,e^{-\frac{t^2}{4(\eta+\ln|y|)}}\,e^{-|y|^2}$$

with $\eta \in (0,1)$ as before, since $\ln|y| > 12$. As a result, we get a bound for I_1 similar to (3.9) but with an extra factor t^{-4-n}, and thus also the bound in (3.1).

Lemma 3.2 is proved. □

Proposition 3.3. *Let f be a measurable function on $\mathbb{R}^n$ satisfying* (1.3). *Then for all $i \in \{1,2,\dots,n\}$ and $x \in \mathbb{R}^n$,*

$$\partial_{x_i}\partial_t P_{s+t}f(x) = \int_{\mathbb{R}^n} \partial_{x_i}P_s(x,y)\,\partial_t P_t f(y)\,dy, \qquad s,\,t>0, \tag{3.11}$$

and

$$\lim_{t\to\infty} \partial_{x_i}P_t f(x) = 0. \tag{3.12}$$

Proof. We can assume $|x| < R$ for some $R > 0$ and thus apply the estimates from Lemma 3.2. First we verify the absolute convergence of the integral in (3.11), by showing that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\partial_{x_i}P_s(x,y)||\partial_t P_t(y,z)||f(z)|\,dy\,dz < \infty.$$

Lemmas 3.2 and 3.1 imply that this integral is, up to a factor $C(n,R)$, no larger than

$$\frac{1+s^{-4-n}}{t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P_{s/2}(x,y)P_{t/2}(y,z)|f(z)|\,dy\,dz = \frac{1+s^{-4-n}}{t}\int_{\mathbb{R}^n} P_{(s+t)/2}(x,z)|f(z)|\,dz < \infty,$$

where the equality comes from the semigroup property. The last integral here is finite because of (1.3); indeed, [1, formula (6.4)] says that $P_t(x,y)$ is controlled by $e^{-|y|^2}/\sqrt{\ln(e+|y|)}$, locally uniformly in x and t.

Our next step consists in integrating the right-hand side of (3.11) along intervals in the variables x_i and t. We choose two points x', $x'' \in \mathbb{R}^n$ with $|x'|$, $|x''| < R$ which differ only in the i:th coordinate, and also two points t', $t'' > 0$. Fubini's theorem applies because of the above estimates, and we get

$$\begin{aligned}
&\int_{x_i'}^{x_i''} \int_{t'}^{t''} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{x_i} P_s(x,y) \partial_t p_t(y,z) f(z)\, dy\, dz \right) dt\, dx_i \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [P_s(x'',y) - P_s(x',y)]\, [P_{t''}(y,z) - P_{t'}(y,z)] f(z)\, dy\, dz \\
&= P_{s+t''} f(x'') - P_{s+t''} f(x') - P_{s+t'} f(x'') + P_{s+t'} f(x').
\end{aligned}$$

From this, we obtain (3.11) by differentiating with respect to x_i'' and t''.

Finally, (3.12) is a direct consequence of (3.2) and (1.3). □

Proposition 3.3 now allows us to apply the method of proof of [4, Proposition 3.2] and obtain the same estimates as there.

Corollary 3.4. *Let $\alpha \in (0,1)$ and let $f \in \mathrm{GGLip}_\alpha$ with norm 1.*

(i) *For all $i \in \{1, 2, \dots, n\}$, $t > 0$ and $x \in \mathbb{R}^n$,*

$$|\partial_{x_i} P_t f(x)| \le C t^{\alpha - 1}.$$

(ii) *For all $t > 0$ and $x = (x_1, 0, \dots, 0) \in \mathbb{R}^n$ with $x_1 \ge 0$,*

$$|\partial_{x_1} P_t f(x)| \le C t^{\alpha - 2} (1 + x_1)^{-1}.$$

4 Proof of Theorem 1.2

(i) $\Longrightarrow$ (ii): We assume that f satisfies (1.3) and (1.1). According to [1, Theorem 1.1], $P_t f(x) \to f(x)$ as $t \to 0$ for a.a. $x \in \mathbb{R}^n$, and thus we can modify f on a null set so that this convergence holds for all x.

Now fix x, $y \in \mathbb{R}^n$. For all $t > 0$, we write

$$|f(x) - f(y)| \le |f(x) - P_t f(x)| + |P_t f(x) - P_t f(y)| + |P_t f(y) - f(y)|\,. \tag{4.1}$$

Using Corollary 3.4 (i) and arguing as in the verification of [4, formula (3.7)], we get

$$|f(x) - f(y)| \lesssim |x - y|^\alpha. \tag{4.2}$$

To obtain (1.5), it is then enough to prove that

$$|f(x) - f(y)| \lesssim d(x, y_x)^{\frac{\alpha}{2}} + |y_x'|^\alpha.$$

By writing

$$|f(x) - f(y)| \le |f(x) - f(y_x)| + |f(y_x) - f(y)|$$

and applying (4.2) to the last term here, we see that we need only verify that

$$|f(x) - f(y_x)| \lesssim d(x, y_x)^{\frac{\alpha}{2}}.$$

Making a rotation, we can assume that $x = (x_1, 0, \ldots, 0)$ with $x_1 \ge 0$ and $y_x = (y_1, 0, \ldots, 0)$.

We estimate $|f(x) - f(y_x)|$ as in (4.1). Of the three terms we then get, the first and third are controlled by t^α. To the second term, we apply Corollary 3.4 (ii) and the one-dimensional integral expression (1.4) for d. As a result,

$$|f(x) - f(y_x)| \lesssim t^\alpha + t^{\alpha-2} d(x, y_x),$$

and here we choose $t = d(x, y_x)^{1/2}$. This leads to (1.5), and the implication (i) $\Longrightarrow$ (ii) is proved.

(ii) $\Longrightarrow$ (i): Letting $y = 0$, we see that (1.5) implies that $f(x) = O((\ln|x|)^{\alpha/2})$ as $|x| \to \infty$ and thus also (1.3). We must verify (1.1).

Using the fact that $\int_{\mathbb{R}^n} \partial_t P_t(x, y)\, dy = 0$ and Lemma 2.2, we can write

$$\begin{aligned} |t\partial_t P_t f(x)| &= \left| \int_{\mathbb{R}^n} t\partial_t P_t(x, y)[f(y) - f(x)]\, dy \right| \\ &\lesssim \int_{\mathbb{R}^n} [K_1(t, x, y) + K_2(t, x, y) + \widetilde{K_3}(t, x, y) + K_4(t, x, y)] |f(y) - f(x)|\, dy. \end{aligned}$$

We thus get four integrals to control by t^α. For $\int_{\mathbb{R}^n} K_1(t, x, y)|f(y) - f(x)|\, dy$, we can apply the same simple argument as in [4, end of Section 3], since it uses only the quantity $|x - y|^\alpha$ in (1.5).

The integral involving $K_2(t, x, y)$ can also be estimated as in [4], because (1.7) applies in the support of $K_2(t, x, y)$.

For the integral with $\widetilde{K_3}(t, x, y)$, we apply the inequality $(a+b)^\kappa \le a^\kappa + b^\kappa$ with $a, b > 0$ and $\kappa = \alpha/2 \in (0, 1)$ to the expression in (1.5) and get

$$\begin{aligned} &\int_{\mathbb{R}^n} \widetilde{K_3}(t, x, y)|f(y) - f(x)|\, dy \\ &\quad \lesssim \min\left\{1, \frac{t}{\sqrt{\ln(e + |x|)}}\right\} \int_{\mathbb{R}^n} \left((\ln(1 + |x|))^{\frac{\alpha}{2}} + (\ln(1 + |y_x|))^{\frac{\alpha}{2}} + |y_x'|^\alpha \right) \exp^*(-|y|^2)\, dy \end{aligned}$$

The minimum here is no larger than $t^\alpha / [\ln(e + |x|)]^{\frac{\alpha}{2}}$, which leads immediately to the bound t^α for the whole expression.

Finally,

$$\begin{aligned} \int_{\mathbb{R}^n} K_4(t, x, y)|f(y) - f(x)|\, dy \le &\int_{\substack{x\cdot y > 0 \\ 1 < |y_x| < |x|/2}} \frac{t}{|y_x|} \left(\ln \frac{|x|}{|y_x|} \right)^{-\frac{3}{2}} \exp^*\left(-\frac{t^2}{\ln \frac{|x|}{|y_x|}} \right) \\ &\times \exp^*(-|y_x'|^2) \left([\ln(1 + |x|) - \ln(1 + |y_x|)]^{\frac{\alpha}{2}} + |y_x'|^\alpha \right) dy \end{aligned} \tag{4.3}$$

When $1 < |y_x| < |x|/2$, we have

$$|\ln(1+|x|) - \ln(1+|y_x|)| = \ln \frac{1+|x|}{1+|y_x|} \simeq \ln \frac{|x|}{|y_x|}.$$

After a rotation, we can assume that $x = (x_1, 0, \dots, 0)$ with $x_1 > 0$, so that $y_x = (y_1, 0, \dots, 0)$ and $y'_x = (0, y')$ and we have $1 < y_1 < x_1/2$. The right-hand integral in (4.3) is bounded by a constant times

$$\int_1^{x_1/2} \int_{\mathbb{R}^{n-1}} \frac{t}{y_1} \left(\ln \frac{x_1}{y_1}\right)^{-\frac{3}{2}} \exp^* \left(-\frac{t^2}{\ln \frac{x_1}{y_1}}\right) \exp^*(-|y'|^2) \left(\left[\ln \frac{x_1}{y_1}\right]^{\frac{\alpha}{2}} + |y'|^\alpha\right) dy'\, dy_1.$$

Integrating in y' and noticing that $\ln(x_1/y_1) \gtrsim 1$, we can control this double integral by

$$\int_1^{x_1/2} \frac{t}{y_1} \left(\ln \frac{x_1}{y_1}\right)^{\frac{\alpha}{2}-\frac{3}{2}} \exp^* \left(-\frac{t^2}{\ln \frac{x_1}{y_1}}\right) dy_1.$$

The transformation of variable $s = t^{-2}(\ln x_1 - \ln y_1)$ now gives the desired bound t^α.

Summing up, we have verified (1.1) and (i). The norm equivalence (1.6) also follows, and this ends the proof of Theorem 1.2.

5 An example of a function in GGLip_α

With $\alpha \in (0,1)$, we consider the function

$$f(x) = [\ln(e+|x|)]^{\alpha/2}, \qquad x \in \mathbb{R}^n.$$

We shall verify that f belongs to GGLip_α, using Theorem 1.2.

The estimate

$$|f(x) - f(y)| \lesssim |x-y|^\alpha \tag{5.1}$$

is easy and left to the reader.

To show that

$$|f(x) - f(y)| \lesssim |\ln(e+|x|) - \operatorname{sgn}\langle x, y\rangle \ln(e+|y_x|)|^{\frac{\alpha}{2}} + |y'_x|^\alpha, \tag{5.2}$$

write

$$|f(x) - f(y)| \le |f(x) - f(y_x)| + |f(y_x) - f(y)|.$$

The last term here is controlled by $|y'_x|^\alpha$, because of (5.1). To the first term on the right, we apply the inequality $|a^\kappa - b^\kappa| \le |a-b|^\kappa$, $a, b > 0$, with $\kappa = \alpha/2 \in (0,1)$, getting

$$|f(x) - f(y_x)| = \left|[\ln(e+|x|)]^{\alpha/2} - [\ln(e+|y_x|)]^{\alpha/2}\right| \le |\ln(e+|x|) - \ln(e+|y_x|)|^{\frac{\alpha}{2}}.$$

This implies (5.2), and it follows that $f \in \mathrm{GGLip}_\alpha$.

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